Left Random Context Grammars

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- **Motivation**
- **Preliminaries and Definitions**
- **Example**
- **Results**
- **Concluding Remarks and Open Problems**

Acknowledgment

This presentation is based on an upcoming paper written jointly with prof. Alexander Meduna.

Motivation

- A natural generalization of left forbidding grammars, introduced in [1].
- Practical viewpoint: we examine only prefixes of sentential forms.
- Theoretical viewpoint: what is the impact of this restriction on the generative power of random context grammars?

Definition

A *type 0 grammar* is a quadruple

$$G = (N, T, P, S),$$

where

- N is the alphabet of *nonterminals*,
- T is the alphabet of *terminals* ($N \cap T = \emptyset$),
- P is a finite set of *rules* of the form

$$x \to y,$$

  where $x \in (N \cup T)^* N (N \cup T)^*$ and $y \in (N \cup T)^*$,
- $S \in N$ is the *starting nonterminal*.

Definition

The relation of a *direct derivation*, denoted by $\Rightarrow$, is defined as follows: if

- $u, v \in (N \cup T)^*$,
- $x \to y \in P$,

then

$$uxv \Rightarrow uyv \text{ in } G.$$

Definition

The *language of G*, denoted by $L(G)$, is defined as

$$L(G) = \{w \in T^* \mid S \Rightarrow^* w\},$$

where $\Rightarrow^*$ is the reflexive and transitive closure of $\Rightarrow$.

Type 1 and Type 2 Grammars

Definition

A type 0 grammar $G = (N, T, P, S)$ is

- a *type 1 grammar (context-sensitive)* if every $x \to y \in P$ satisfies

$$|x| \leq |y|$$

- a *type 2 grammar (context-free)* if every $x \to y \in P$ satisfies

$$x \in N$$

Definition

A *random context grammar* (see (5)) is a quadruple

$$G = (N, T, P, S),$$

where

- N, T, and S are defined as in a context-free grammar,
- P is a finite set of *rules* of the form

$$\lfloor A \rightarrow x, U, W \rfloor$$

where $A \in N$, $x \in (N \cup T)^*$, and $U, W \subseteq N$.

U … *permitting context*

W … *forbidding context*

Definition

The relation of a *direct derivation*, denoted by $\Rightarrow$, is defined as follows: if

- $u, v \in (N \cup T)^*$,
- $\lfloor A \rightarrow x, U, W \rfloor \in P$,
- $U \subseteq alph(uAv)$,
- $W \cap alph(uAv) = \emptyset$,

then

$$uAv \Rightarrow uxv \text{ in } G.$$

Definition

The *language of G*, denoted by $L(G)$, is defined as

$$L(G) = \{w \in T^* \mid S \Rightarrow^* w\},$$

where $\Rightarrow^*$ is the reflexive and transitive closure of $\Rightarrow$.

Random Context Grammars (cont.)

Definition

If $\lfloor A \rightarrow x, U, W \rfloor \in P$ implies $|x| \geq 1$, then G is a *propagating random context grammar*.

Definition

If $\lfloor A \rightarrow x, U, W \rfloor \in P$ implies $W = \emptyset$, then G is a *permitting grammar*.

Definition

If $\lfloor A \rightarrow x, U, W \rfloor \in P$ implies $U = \emptyset$, then G is a *forbidding grammar*.

Definition

Let $G = (N, T, P, S)$ be a random context grammar. G is reffered to as a *left random context grammar* if its relation of a direct derivation ($\Rightarrow$) is defined as follows: if

- $u, v \in (N \cup T)^*$,
- $\lfloor A \rightarrow x, U, W \rfloor \in P$,
- $U \subseteq alph(u)$, (not uAv !)
- $W \cap alph(u) = \emptyset$, (not uAv !)

then

$$uAv \Rightarrow uxv \text{ in } G.$$

Definition

If $\lfloor A \to x, U, W \rfloor \in P$ implies $|x| \geq 1$, then G is a *propagating left random context grammar*.

Definition

If $\lfloor A \to x, U, W \rfloor \in P$ implies $W = \emptyset$, then G is a *left permitting grammar*.

Definition

If $\lfloor A \to x, U, W \rfloor \in P$ implies $U = \emptyset$, then G is a *left forbidding grammar*.

Denotation of Language Families

- $\mathscr{L}_{CF}$ ... the family of context-free languages
- $\mathscr{L}_{CS}$ ... the family of context-sensitive languages
- $\mathscr{L}_{RE}$ ... the family of recursively enumerable languages
- $\mathscr{L}_{SC}$ ... the family of propagating scattered context languages

Denotation of Language Families (cont.)

- $\mathscr{L}_{RC}^{\varepsilon}$ ... the family of languages generated by random context grammars
- $\mathscr{L}_{P}^{\varepsilon}$ ... the family of languages generated by permitting grammars
- $\mathscr{L}_{F}^{\varepsilon}$ ... the family of languages generated by forbidding grammars
- $\mathscr{L}_{RC}$ ... the family of languages generated by propagating random context grammars
- $\mathscr{L}_{P}$ ... the family of languages generated by propagating permitting grammars
- $\mathscr{L}_{F}$ ... the family of languages generated by propagating forbidding grammars

Denotation of Language Families (cont.)

- $\mathscr{L}^{\varepsilon}_{LRC}$ . . . the family of languages generated by left random context grammars
- $\mathscr{L}^{\varepsilon}_{LP}$ . . . the family of languages generated by left permitting grammars
- $\mathscr{L}^{\varepsilon}_{LF}$ . . . the family of languages generated by left forbidding grammars
- $\mathscr{L}_{LRC}$ . . . the family of languages generated by propagating left random context grammars
- $\mathscr{L}_{LP}$ . . . the family of languages generated by propagating left permitting grammars
- $\mathscr{L}_{LF}$ . . . the family of languages generated by propagating left forbidding grammars

Example

Consider $K = \{a^n b^m c^m \mid 1 \leq m \leq n\}$. This non-context-free language is generated by the left random context grammar G defined as

$$G = (\{S, A, B, X, Y\}, \{a, b, c\}, P, S)$$

with P containing the following seven rules:

$$\lfloor S \to AX, \emptyset, \emptyset \rfloor, \quad \lfloor X \to bc, \emptyset, \emptyset \rfloor,$$
$$\lfloor A \to a, \emptyset, \emptyset \rfloor, \quad \lfloor X \to bYc, \{B\}, \emptyset \rfloor,$$
$$\lfloor A \to aB, \emptyset, \emptyset \rfloor, \quad \lfloor Y \to X, \{A\}, \emptyset \rfloor.$$
$$\lfloor B \to A, \emptyset, \emptyset \rfloor,$$

Notice that G is, in fact, a propagating left permitting grammar.

Example

P : $\lfloor S \rightarrow AX, \emptyset, \emptyset \rfloor$,
$\lfloor A \rightarrow a, \emptyset, \emptyset \rfloor$,
$\lfloor A \rightarrow aB, \emptyset, \emptyset \rfloor$,
$\lfloor B \rightarrow A, \emptyset, \emptyset \rfloor$,
$\lfloor X \rightarrow bc, \emptyset, \emptyset \rfloor$,
$\lfloor X \rightarrow bYc, \{B\}, \emptyset \rfloor$,
$\lfloor Y \rightarrow X, \{A\}, \emptyset \rfloor$.

Observations:

- Every derivation starts with the application of $\lfloor S \rightarrow AX, \emptyset, \emptyset \rfloor$.

Example

P : $\lfloor S \rightarrow AX, \emptyset, \emptyset \rfloor$, $\lfloor X \rightarrow bc, \emptyset, \emptyset \rfloor$,
$\lfloor A \rightarrow a, \emptyset, \emptyset \rfloor$, $\lfloor X \rightarrow bYc, \{B\}, \emptyset \rfloor$,
$\lfloor A \rightarrow aB, \emptyset, \emptyset \rfloor$, $\lfloor Y \rightarrow X, \{A\}, \emptyset \rfloor$.
$\lfloor B \rightarrow A, \emptyset, \emptyset \rfloor$,

Observations:

- Every derivation starts with the application of $\lfloor S \rightarrow AX, \emptyset, \emptyset \rfloor$.
- $\lfloor X \rightarrow bYc, \{B\}, \emptyset \rfloor$ is applicable if B, produced by $\lfloor A \rightarrow aB, \emptyset, \emptyset \rfloor$, occurs to the left of X in the current sentential form.

Example

P : $\lfloor S \rightarrow AX, \emptyset, \emptyset \rfloor$,
$\lfloor A \rightarrow a, \emptyset, \emptyset \rfloor$,
$\lfloor A \rightarrow aB, \emptyset, \emptyset \rfloor$,
$\lfloor B \rightarrow A, \emptyset, \emptyset \rfloor$,
$\lfloor X \rightarrow bc, \emptyset, \emptyset \rfloor$,
$\lfloor X \rightarrow bYc, \{B\}, \emptyset \rfloor$,
$\lfloor Y \rightarrow X, \{A\}, \emptyset \rfloor$.

Observations:

- Every derivation starts with the application of $\lfloor S \rightarrow AX, \emptyset, \emptyset \rfloor$.
- $\lfloor X \rightarrow bYc, \{B\}, \emptyset \rfloor$ is applicable if B, produced by $\lfloor A \rightarrow aB, \emptyset, \emptyset \rfloor$, occurs to the left of X in the current sentential form.
- Similarly, $\lfloor Y \rightarrow X, \{A\}, \emptyset \rfloor$ is applicable if A, produced by $\lfloor B \rightarrow A, \emptyset, \emptyset \rfloor$, occurs to the left of Y in the current sentential form.

Example

P : $\lfloor S \rightarrow AX, \emptyset, \emptyset \rfloor$, $\lfloor A \rightarrow a, \emptyset, \emptyset \rfloor$, $\lfloor A \rightarrow aB, \emptyset, \emptyset \rfloor$, $\lfloor B \rightarrow A, \emptyset, \emptyset \rfloor$, $\lfloor X \rightarrow bc, \emptyset, \emptyset \rfloor$, $\lfloor X \rightarrow bYc, \{B\}, \emptyset \rfloor$, $\lfloor Y \rightarrow X, \{A\}, \emptyset \rfloor$.

Observations:

- Every derivation starts with the application of $\lfloor S \rightarrow AX, \emptyset, \emptyset \rfloor$.
- $\lfloor X \rightarrow bYc, \{B\}, \emptyset \rfloor$ is applicable if B, produced by $\lfloor A \rightarrow aB, \emptyset, \emptyset \rfloor$, occurs to the left of X in the current sentential form.
- Similarly, $\lfloor Y \rightarrow X, \{A\}, \emptyset \rfloor$ is applicable if A, produced by $\lfloor B \rightarrow A, \emptyset, \emptyset \rfloor$, occurs to the left of Y in the current sentential form.
- After $\lfloor A \rightarrow a, \emptyset, \emptyset \rfloor$ is applied, only one b and one c can be generated.

Example

P : $\lfloor S \rightarrow AX, \emptyset, \emptyset \rfloor$, $\lfloor A \rightarrow a, \emptyset, \emptyset \rfloor$, $\lfloor A \rightarrow aB, \emptyset, \emptyset \rfloor$, $\lfloor B \rightarrow A, \emptyset, \emptyset \rfloor$, $\lfloor X \rightarrow bc, \emptyset, \emptyset \rfloor$, $\lfloor X \rightarrow bYc, \{B\}, \emptyset \rfloor$, $\lfloor Y \rightarrow X, \{A\}, \emptyset \rfloor$.

Every derivation that generates $w \in L(G)$ is of the form

$$
\begin{aligned}
S & \Rightarrow AX \\
& \Rightarrow^* a^u AX \\
& \Rightarrow a^{u+1} BX \\
& \Rightarrow a^{u+1} BbYc \\
& \Rightarrow a^{u+1} AbYc \\
& \Rightarrow^* a^{u+1+v} AbYc \\
& \Rightarrow a^{u+1+v} AbXc \\
& \dots \\
& \Rightarrow^* a^n Ab^m Xc^m \\
& \Rightarrow^2 a^{n+1} b^{m+1} c^{m+1} = w.
\end{aligned}
$$

Hence, $L(G) = K$.

Notice

As was demonstrated in my talk, the proofs of the last two theorems in the following slide are not correct.

Next, we prove the following four inclusions (identities).

Theorem

$\mathscr{L}_{LF}^{\varepsilon} = \mathscr{L}_{LF} = \mathscr{L}_{CF}$

Theorem

$\mathscr{L}_{CF} \subset \mathscr{L}_{LP} \subseteq \mathscr{L}_{SC}$

Theorem

$\mathscr{L}_{LRC} = \mathscr{L}_{CS}$

Theorem

$\mathscr{L}_{LRC}^{\varepsilon} = \mathscr{L}_{RE}$

The following relations regarding $\mathscr{L}_{LF}^{\varepsilon}$ and $\mathscr{L}_{LF}$ were established in [1].

Lemma

$\mathscr{L}_{CF} \subseteq \mathscr{L}_{LF}^{\varepsilon}$

Proof (idea): Follows from the definition of a left forbidding grammar. $\square$

Lemma

$\mathscr{L}_{LF}^{\varepsilon} \subseteq \mathscr{L}_{CF}$

Proof (idea): Let $G = (N, T, P, S)$ be a left forbidding grammar. Define the context free grammar

$$H = (N, T, P', S)$$

with $P' = \{A \to x \mid \lfloor A \to x, \emptyset, W \rfloor \in P\}$.

Lemma

$\mathscr{L}_{LF}^{\varepsilon} \subseteq \mathscr{L}_{CF}$

Proof (idea): Let $G = (N, T, P, S)$ be a left forbidding grammar. Define the context free grammar

$$H = (N, T, P', S)$$

with $P' = \{A \to x \mid \lfloor A \to x, \emptyset, W \rfloor \in P\}$.

- $L(G) \subseteq L(H)$

Lemma

$\mathscr{L}_{LF}^{\varepsilon} \subseteq \mathscr{L}_{CF}$

Proof (idea): Let $G = (N, T, P, S)$ be a left forbidding grammar. Define the context free grammar

$$H = (N, T, P', S)$$

with $P' = \{A \rightarrow x \mid \lfloor A \rightarrow x, \emptyset, W \rfloor \in P\}$.

- $L(G) \subseteq L(H)$ . . . Any successful derivation of G is also a successful derivation of H.

Lemma

$\mathscr{L}_{LF}^{\varepsilon} \subseteq \mathscr{L}_{CF}$

Proof (idea): Let $G = (N, T, P, S)$ be a left forbidding grammar. Define the context free grammar

$$H = (N, T, P', S)$$

with $P' = \{A \to x \mid \lfloor A \to x, \emptyset, W \rfloor \in P\}$.

- $L(G) \subseteq L(H)$ . . . Any successful derivation of G is also a successful derivation of H.
- $L(H) \subseteq L(G)$

Lemma

$\mathscr{L}_{LF}^{\varepsilon} \subseteq \mathscr{L}_{CF}$

Proof (idea): Let $G = (N, T, P, S)$ be a left forbidding grammar. Define the context free grammar

$$H = (N, T, P', S)$$

with $P' = \{A \rightarrow x \mid \lfloor A \rightarrow x, \emptyset, W \rfloor \in P\}$.

- $L(G) \subseteq L(H)$ ... Any successful derivation of G is also a successful derivation of H.
- $L(H) \subseteq L(G)$ ... Let $w \in L(H)$ be derived using a leftmost derivation. Such a leftmost derivation is also possible in G because the leftmost nonterminal can always be rewritten.

Consequently, $L(H) = L(G)$. □

Theorem

$\mathscr{L}_{LF}^{\varepsilon} = \mathscr{L}_{CF}$

Proof: Follows directly from the two previous lemmas. □

As an immediate consequence of this theorem, we have that erasing rules can be eliminated from any left forbidding grammar.

Corollary

$\mathscr{L}_{LF}^{\varepsilon} = \mathscr{L}_{LF} = \mathscr{L}_{CF}$

Theorem

$\mathscr{L}_{CF} \subset \mathscr{L}_{LP}$

Proof:

- $\mathscr{L}_{CF} \subseteq \mathscr{L}_{LP}$

Theorem

$\mathscr{L}_{CF} \subset \mathscr{L}_{LP}$

Proof:

- $\mathscr{L}_{CF} \subseteq \mathscr{L}_{LP}$ . . . Follows from the definition of a left permitting grammar.

Towards $\mathscr{L}_{CF} \subset \mathscr{L}_{LP} \subseteq \mathscr{L}_{SC}$

Theorem

$\mathscr{L}_{CF} \subset \mathscr{L}_{LP}$

Proof:

- $\mathscr{L}_{CF} \subseteq \mathscr{L}_{LP}$ . . . Follows from the definition of a left permitting grammar.
- $\mathscr{L}_{CF} \subset \mathscr{L}_{LP}$

Theorem

$\mathscr{L}_{CF} \subset \mathscr{L}_{LP}$

Proof:

- $\mathscr{L}_{CF} \subseteq \mathscr{L}_{LP}$ ... Follows from the definition of a left permitting grammar.
- $\mathscr{L}_{CF} \subset \mathscr{L}_{LP}$ ... Follows from the fact that the non-context-free language $\{a^n b^m c^m \mid 1 \leq m \leq n\}$ can be generated by the propagating left permitting grammar G from the example. □

Definition

A *propagating scattered context grammar* (see (2)) is a quadruple,

$$G = (N, T, P, S),$$

where

- N, T, and S are defined as in a context-free grammar;
- P is a finite set of rules of the form

$$(A_1, A_2, \dots, A_n) \rightarrow (x_1, x_2, \dots, x_n),$$

where $A_i \in N$ and $x_i \in (N \cup T)^+$, for all i.

Definition

The relation of a *direct derivation*, denoted by $\Rightarrow$, is defined as follows: if

- $(A_1, A_2, \ldots, A_n) \rightarrow (x_1, x_2, \ldots, x_n) \in P$,
- $u = u_1 A_1 u_2 A_2 \cdots u_n A_n u_{n+1}$, and
- $v = u_1 x_1 \; u_2 x_2 \; \cdots u_n x_n \; u_{n+1}$,

where $u_i \in (N \cup T)^*$, for all i, then

$$u \Rightarrow v \text{ in } G.$$

Definition

The *language of G*, denoted by $L(G)$, is defined as

$$L(G) = \{w \in T^+ \mid S \Rightarrow^* w\},$$

where $\Rightarrow^*$ is the reflexive and transitive closure of $\Rightarrow$.

Theorem

$\mathscr{L}_{LP} \subseteq \mathscr{L}_{SC}$

Proof (idea): Let $G = (N, T, P, S)$ be a propagating left permitting grammar. Define the propagating scattered context grammar

$$H = (N, T, P', S)$$

with P' constructed as follows:

1. for every $\lfloor A \rightarrow x, \emptyset, \emptyset \rfloor \in P$, add $(A) \rightarrow (x)$ to P';
2. for every $\lfloor A \rightarrow x, \{X_1, X_2, \dots, X_n\}, \emptyset \rfloor \in P$ and every permutation $(i_1, i_2, \dots, i_n)$ of $(1, 2, \dots, n)$, where $n \geq 1$, add $(X_{i_1}, X_{i_2}, \dots, X_{i_n}, A) \rightarrow (X_{i_1}, X_{i_2}, \dots, X_{i_n}, x)$ to P'.

It can be easily shown that $L(G) = L(H)$. □

Lemma

$\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$

Proof (idea): We show how to simulate any context-sensitive grammar in the Penttonen normal form (see (3)) by a propagating left random context grammar.

- Let $G = (N, T, P, S)$ be a context-sensitive grammar in the Penttonen normal form.
- We next construct a propagating left random context grammar H such that $L(H) = L(G)$.

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

- Set $\bar{N} = \{\bar{A} \mid A \in N\}$,
  $\hat{N} = \{\hat{A} \mid A \in N\}$,
  $N' = N \cup \bar{N} \cup \hat{N}$.
- Define the propagating left random context grammar

$$H = (N', T, P', S)$$

  with P' constructed as follows:

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ *(cont.):*

Simulation of $A \rightarrow a \in P$, $A \in N$, $a \in T$:

- add $\lfloor A \rightarrow a, \emptyset, N' \rfloor$ to P'.

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

Simulation of $A \rightarrow a \in P$, $A \in N$, $a \in T$:

- add $\lfloor A \rightarrow a, \emptyset, N' \rfloor$ to P'.

Simulation of $A \rightarrow BC \in P$, $A, B, C \in N$:

- add $\lfloor A \rightarrow BC, \emptyset, \emptyset \rfloor$ to P'.

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

Simulation of $A \rightarrow a \in P$, $A \in N$, $a \in T$:

- add $\lfloor A \rightarrow a, \emptyset, N' \rfloor$ to P'.

Simulation of $A \rightarrow BC \in P$, $A, B, C \in N$:

- add $\lfloor A \rightarrow BC, \emptyset, \emptyset \rfloor$ to P'.

Simulation of $AB \rightarrow AC \in P$, $A, B, C \in N$:

- the tricky part...

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ *(cont.):*

Simulation of $AB \rightarrow AC \in P$ (example):

$aDbEABc$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

Simulation of $AB \rightarrow AC \in P$ (example):

$aDbEABc$

Towards $\mathscr{L}_{LRC} = \mathscr{L}_{CS}$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

Simulation of $AB \rightarrow AC \in P$ (example):

$$aDbEABc \quad \Rightarrow \quad a\overline{D}bEABc \qquad \lfloor D \rightarrow \overline{D}, \emptyset, N \cup \hat{N} \rfloor$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

Simulation of $AB \rightarrow AC \in P$ (example):

$$aDbEABc \quad \Rightarrow \quad a\overline{D}bEABc \qquad \lfloor D \rightarrow \overline{D}, \emptyset, N \cup \hat{N} \rfloor$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

Simulation of $AB \rightarrow AC \in P$ (example):

$$
\begin{array}{lcll}
aDbEABc & \Rightarrow & a\overline{D}b\color{red}{E}\color{black}ABc & \lfloor D \rightarrow \overline{D}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\color{red}{\overline{E}}\color{black}ABc & \lfloor E \rightarrow \overline{E}, \emptyset, N \cup \hat{N} \rfloor
\end{array}
$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

Simulation of $AB \to AC \in P$ (example):

$$\begin{array}{lcll} aDbEABc & \Rightarrow & a\overline{D}bEABc & \lfloor D \to \overline{D}, \emptyset, N \cup \hat{N} \rfloor \\ & \Rightarrow & a\overline{D}b\overline{E}ABc & \lfloor E \to \overline{E}, \emptyset, N \cup \hat{N} \rfloor \end{array}$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ *(cont.):*

Simulation of $AB \rightarrow AC \in P$ (example):

$$
\begin{array}{llll}
aDbEABc & \Rightarrow & a\overline{D}bEABc & \lfloor D \rightarrow \overline{D}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ABc & \lfloor E \rightarrow \overline{E}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Bc & \lfloor A \rightarrow \hat{A}, \emptyset, N \cup \hat{N} \rfloor
\end{array}
$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

Simulation of $AB \rightarrow AC \in P$ (example):

$$\begin{array}{llll} aDbEABc & \Rightarrow & a\overline{D}bEABc & \lfloor D \rightarrow \overline{D}, \emptyset, N \cup \hat{N} \rfloor \\ & \Rightarrow & a\overline{D}b\overline{E}ABc & \lfloor E \rightarrow \overline{E}, \emptyset, N \cup \hat{N} \rfloor \\ & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Bc & \lfloor A \rightarrow \hat{A}, \emptyset, N \cup \hat{N} \rfloor \end{array}$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ *(cont.):*

Simulation of $AB \to AC \in P$ (example):

$$
\begin{array}{llll}
aDbEABc & \Rightarrow & a\overline{D}bEABc & \lfloor D \to \overline{D}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ABc & \lfloor E \to \overline{E}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Bc & \lfloor A \to \hat{A}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Cc & \lfloor B \to C, \{\hat{A}\}, N \rfloor
\end{array}
$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ *(cont.):*

Simulation of $AB \rightarrow AC \in P$ (example):

$$
\begin{array}{llll}
aDbEABc & \Rightarrow & a\overline{D}bEABc & \lfloor D \rightarrow \overline{D}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ABc & \lfloor E \rightarrow \overline{E}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Bc & \lfloor A \rightarrow \hat{A}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Cc & \lfloor B \rightarrow C, \{\hat{A}\}, N \rfloor
\end{array}
$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

Simulation of $AB \to AC \in P$ (example):

$$
\begin{array}{lll}
aDbEABc & \Rightarrow & a\overline{D}bEABc \quad \lfloor D \to \overline{D}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ABc \quad \lfloor E \to \overline{E}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Bc \quad \lfloor A \to \hat{A}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Cc \quad \lfloor B \to C, \{\hat{A}\}, N \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ACc \quad \lfloor \hat{A} \to A, \emptyset, \emptyset \rfloor
\end{array}
$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

Simulation of $AB \rightarrow AC \in P$ (example):

$$
\begin{array}{lcll}
aDbEABc & \Rightarrow & a\overline{D}bEABc & \lfloor D \rightarrow \overline{D}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ABc & \lfloor E \rightarrow \overline{E}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Bc & \lfloor A \rightarrow \hat{A}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Cc & \lfloor B \rightarrow C, \{\hat{A}\}, N \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ACc & \lfloor \hat{A} \rightarrow A, \emptyset, \emptyset \rfloor
\end{array}
$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

Simulation of $AB \to AC \in P$ (example):

$$
\begin{array}{lcll}
aDbEABc & \Rightarrow & a\overline{D}bEABc & \lfloor D \to \overline{D}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ABc & \lfloor E \to \overline{E}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Bc & \lfloor A \to \hat{A}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Cc & \lfloor B \to C, \{\hat{A}\}, N \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ACc & \lfloor \hat{A} \to A, \emptyset, \emptyset \rfloor \\
 & \Rightarrow & a\overline{D}bEACc & \lfloor \overline{E} \to E, \emptyset, \emptyset \rfloor
\end{array}
$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ *(cont.):*

Simulation of $AB \to AC \in P$ (example):

$$
\begin{array}{llll}
aDbEABc & \Rightarrow & a\overline{D}bEABc & \lfloor D \to \overline{D}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ABc & \lfloor E \to \overline{E}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Bc & \lfloor A \to \hat{A}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Cc & \lfloor B \to C, \{\hat{A}\}, N \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ACc & \lfloor \hat{A} \to A, \emptyset, \emptyset \rfloor \\
 & \Rightarrow & a\overline{D}bEACc & \lfloor \overline{E} \to E, \emptyset, \emptyset \rfloor
\end{array}
$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ *(cont.):*

Simulation of $AB \rightarrow AC \in P$ (example):

$$
\begin{array}{llll}
aDbEABc & \Rightarrow & a\overline{D}bEABc & \lfloor D \rightarrow \overline{D}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ABc & \lfloor E \rightarrow \overline{E}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Bc & \lfloor A \rightarrow \hat{A}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Cc & \lfloor B \rightarrow C, \{\hat{A}\}, N \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ACc & \lfloor \hat{A} \rightarrow A, \emptyset, \emptyset \rfloor \\
 & \Rightarrow & a\overline{D}bEACc & \lfloor \overline{E} \rightarrow E, \emptyset, \emptyset \rfloor \\
 & \Rightarrow & aDbEACc & \lfloor \overline{D} \rightarrow D, \emptyset, \emptyset \rfloor
\end{array}
$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

Simulation of $AB \rightarrow AC \in P$ (example):

$$
\begin{array}{llll}
aDbEABc & \Rightarrow & a\overline{D}bEABc & \lfloor D \rightarrow \overline{D}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ABc & \lfloor E \rightarrow \overline{E}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Bc & \lfloor A \rightarrow \hat{A}, \emptyset, N \cup \hat{N} \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}\hat{A}Cc & \lfloor B \rightarrow C, \{\hat{A}\}, N \rfloor \\
 & \Rightarrow & a\overline{D}b\overline{E}ACc & \lfloor \hat{A} \rightarrow A, \emptyset, \emptyset \rfloor \\
 & \Rightarrow & a\overline{D}bEACc & \lfloor \overline{E} \rightarrow E, \emptyset, \emptyset \rfloor \\
 & \Rightarrow & aDbEACc & \lfloor \overline{D} \rightarrow D, \emptyset, \emptyset \rfloor
\end{array}
$$

Proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$ (cont.):

Simulation of $AB \rightarrow AC \in P$, $A, B, C \in N$:

- for every $D \in N$:
  - add $\lfloor D \rightarrow \overline{D}, \emptyset, N \cup \hat{N} \rfloor$ and $\lfloor D \rightarrow \hat{D}, \emptyset, N \cup \hat{N} \rfloor$ to P';
  - add $\lfloor \overline{D} \rightarrow D, \emptyset, \emptyset \rfloor$ and $\lfloor \hat{D} \rightarrow D, \emptyset, \emptyset \rfloor$ to P'.
- add $\lfloor B \rightarrow C, \{\hat{A}\}, N \rfloor$ to P'.

$L(G) = L(H)$ is proved by induction on the lengths of derivations. □

Lemma

$\mathscr{L}_{LRC} \subseteq \mathscr{L}_{CS}$

Proof (idea): Follows from the Workspace theorem (see (4)). $\square$

Theorem

$\mathscr{L}_{LRC} = \mathscr{L}_{CS}$

Proof: Follows directly from the two previous lemmas. □

Lemma

$\mathscr{L}^{\varepsilon}_{LRC} \subseteq \mathscr{L}_{RE}$

Proof: Follows from the Church-Turing thesis. □

Lemma

$\mathscr{L}_{RE} \subseteq \mathscr{L}^{\varepsilon}_{LRC}$

Proof (idea): This lemma can be established by analogy with the proof of $\mathscr{L}_{CS} \subseteq \mathscr{L}_{LRC}$; recall that there is also Penttonen normal form for type 0 grammars (see (3)). □

Theorem

$\mathscr{L}_{LRC}^{\varepsilon} = \mathscr{L}_{RE}$

Proof: Follows directly from the two previous lemmas. □

Relationship of random context grammars and left random context grammars.

Theorem

1. $\mathscr{L}_{RC} \subset \mathscr{L}_{LRC} \subset \mathscr{L}_{RC}^{\varepsilon} = \mathscr{L}_{LRC}^{\varepsilon}$
2. $\mathscr{L}_{LF} = \mathscr{L}_{LF}^{\varepsilon} \subset \mathscr{L}_{F} \subseteq \mathscr{L}_{F}^{\varepsilon}$

Known hierarchy results.

Theorem

1. $\mathscr{L}_{CF} \subset \mathscr{L}_{P} \subset \mathscr{L}_{RC} \subset \mathscr{L}_{LRC} = \mathscr{L}_{CS} \subset \mathscr{L}_{RC}^{\varepsilon} = \mathscr{L}_{LRC}^{\varepsilon} = \mathscr{L}_{RE}$
2. $\mathscr{L}_{CF} = \mathscr{L}_{LF} = \mathscr{L}_{LF}^{\varepsilon} \subset \mathscr{L}_{F} \subseteq \mathscr{L}_{F}^{\varepsilon} \subset \mathscr{L}_{RE}$
3. $\mathscr{L}_{CF} \subset \mathscr{L}_{P} = \mathscr{L}_{P}^{\varepsilon} \subset \mathscr{L}_{CS}$
4. $\mathscr{L}_{CF} \subset \mathscr{L}_{LP} \subseteq \mathscr{L}_{SC} \subseteq \mathscr{L}_{CS}$

Open Problems

1. Establish relations between $\mathscr{L}_P$, $\mathscr{L}_{LP}$, and $\mathscr{L}_{LP}^{\varepsilon}$.
2. Recall that $\mathscr{L}_P = \mathscr{L}_P^{\varepsilon}$ (see (6)). Is it true that $\mathscr{L}_{LP} = \mathscr{L}_{LP}^{\varepsilon}$?
3. Does $\mathscr{L}_{LP} = \mathscr{L}_{LRC}$ hold? If so, then our results would imply $\mathscr{L}_{SC} = \mathscr{L}_{CS}$ and, thereby, solve a longstanding open question.

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Thank you for your attention!