Jumping Finite Automata

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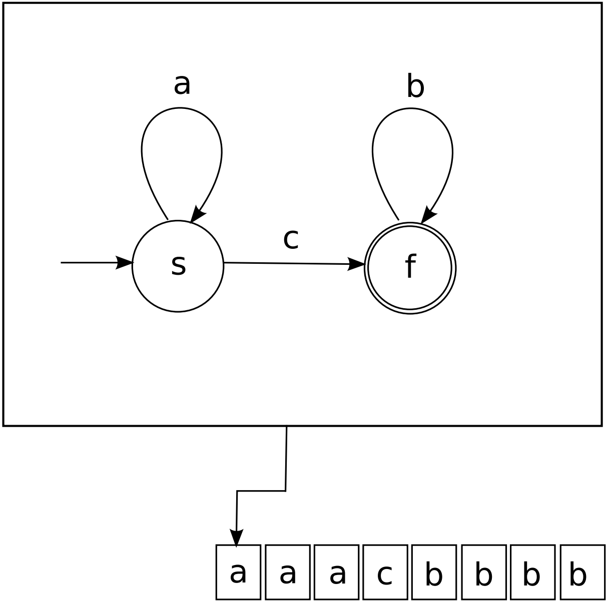
Finite Automata



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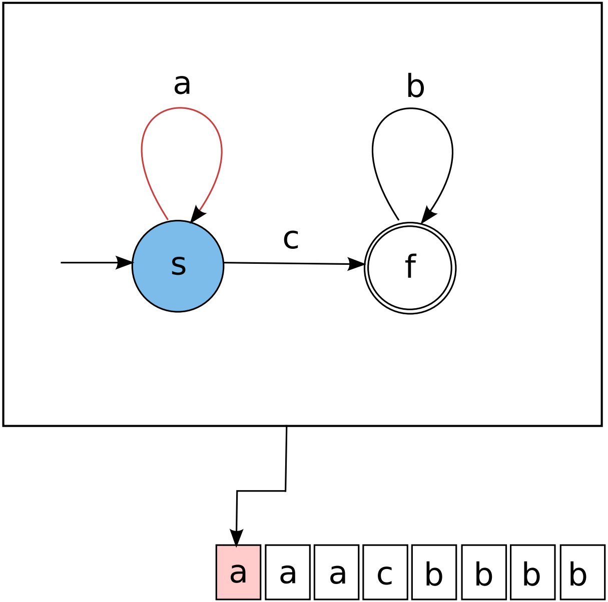

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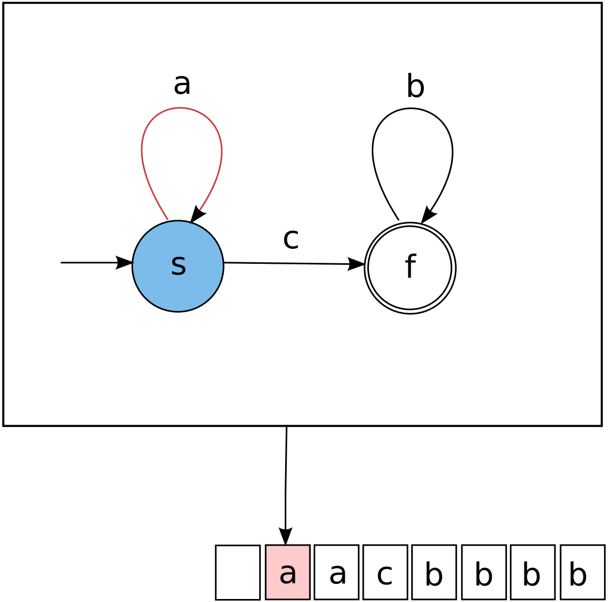

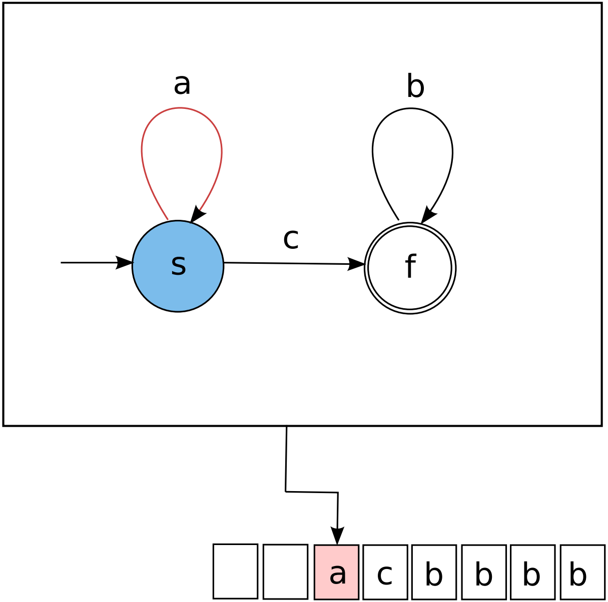
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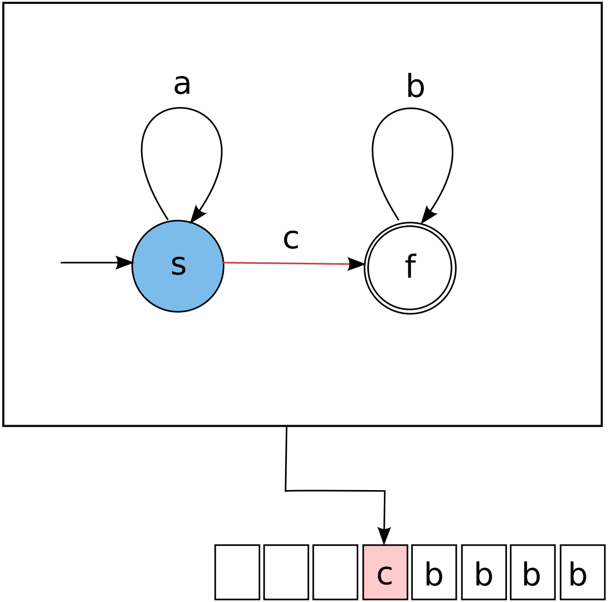
Finite Automata



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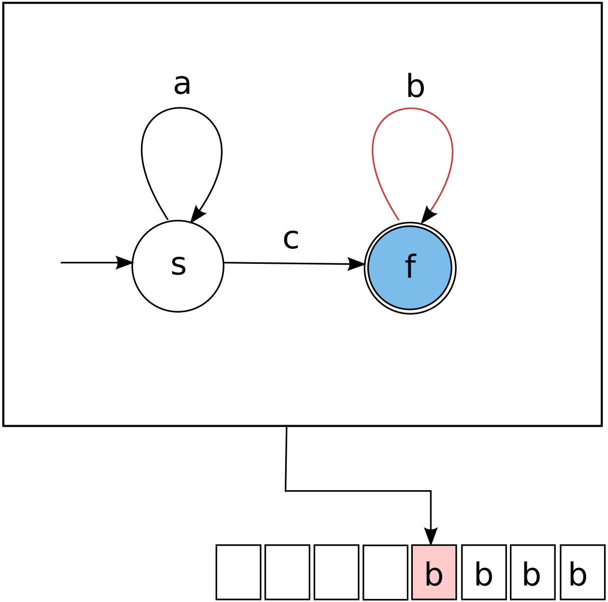

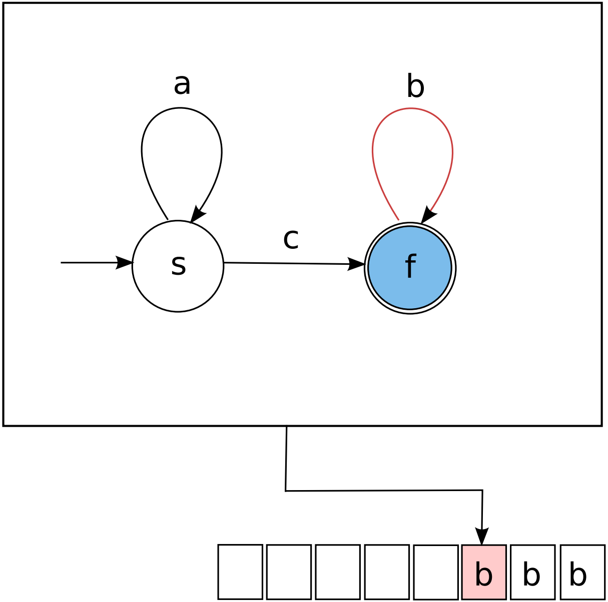
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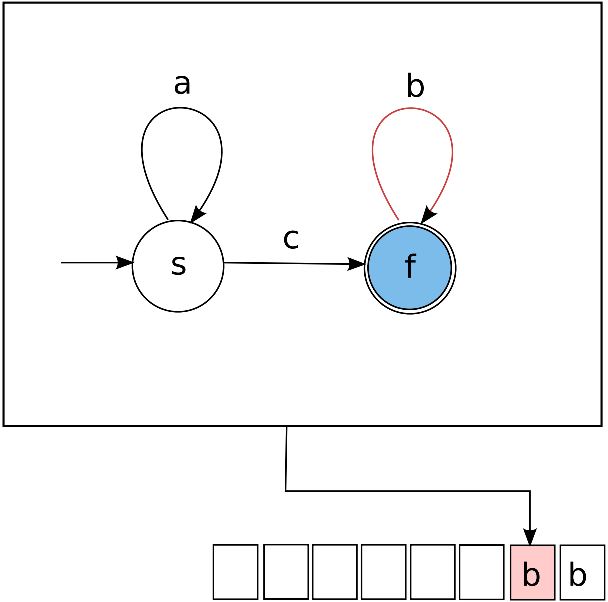
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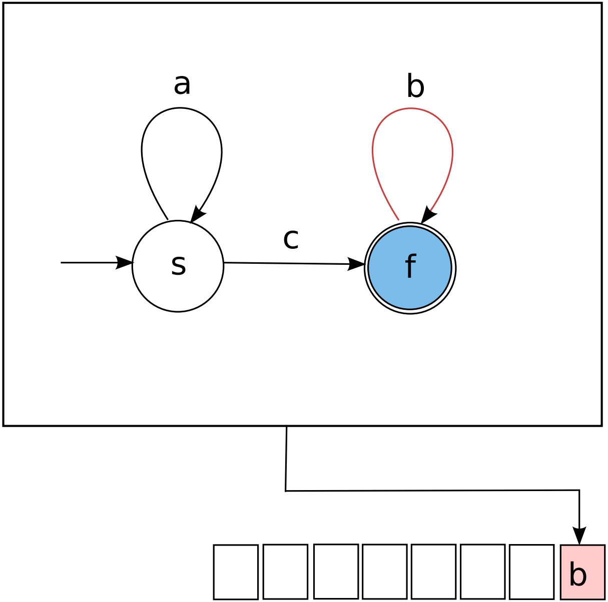
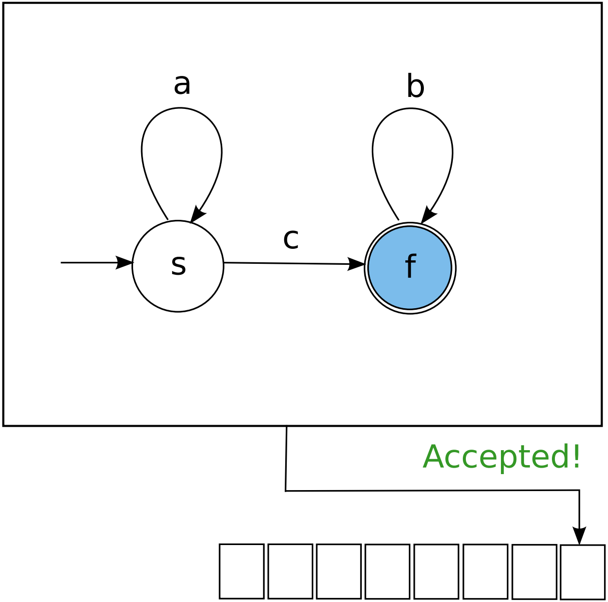
Finite Automata

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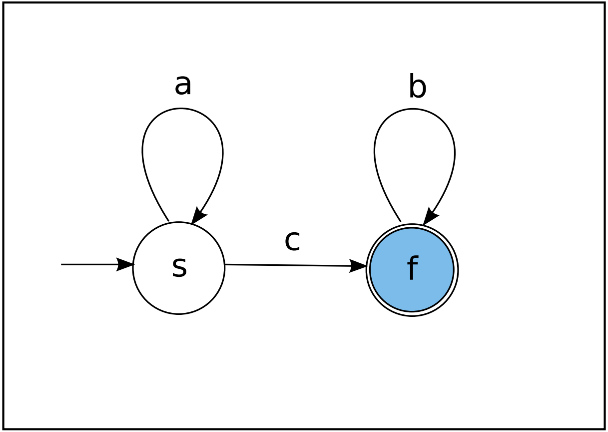
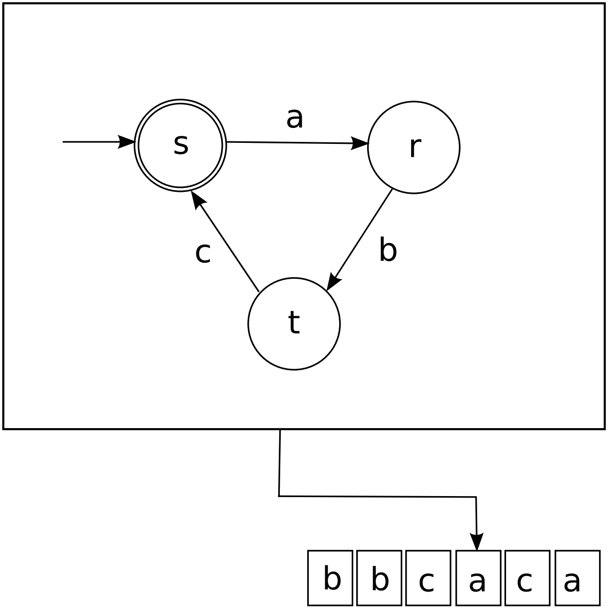
Accepted!

Finite Automata



Accepted language: $\{a\}^*\{c\}\{b\}^*$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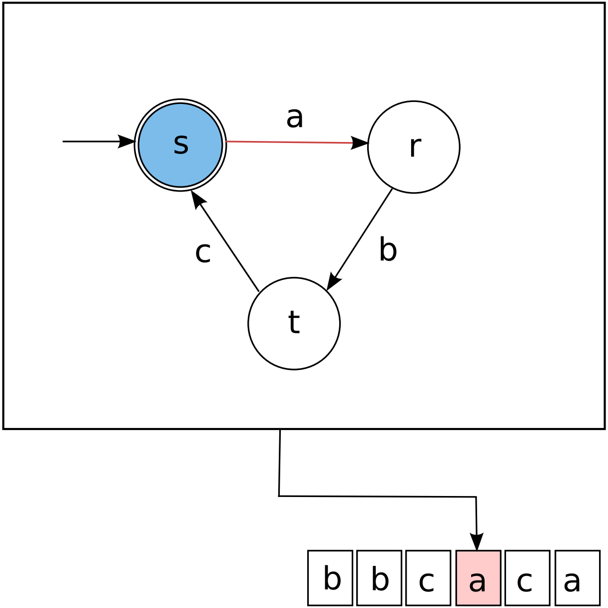
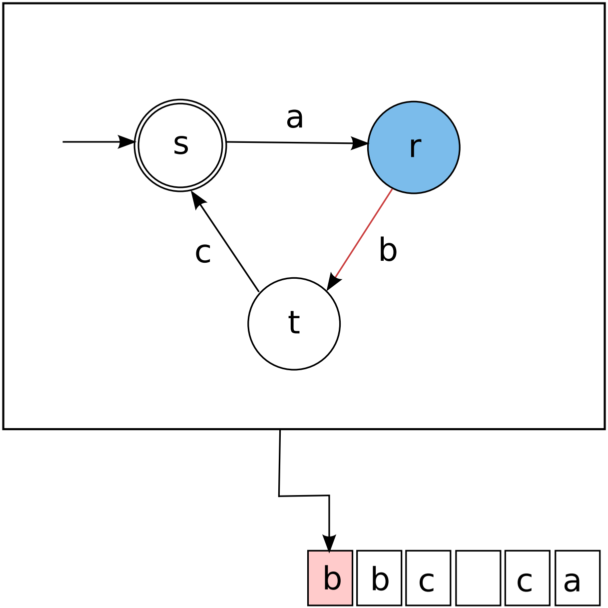
Jumping Finite Automata

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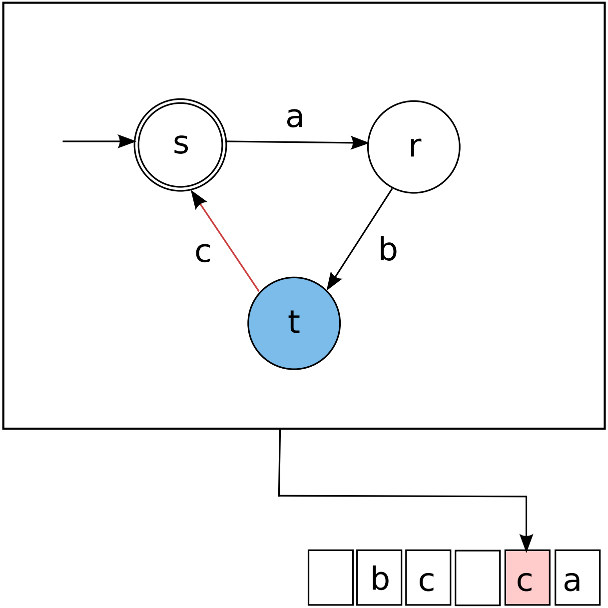
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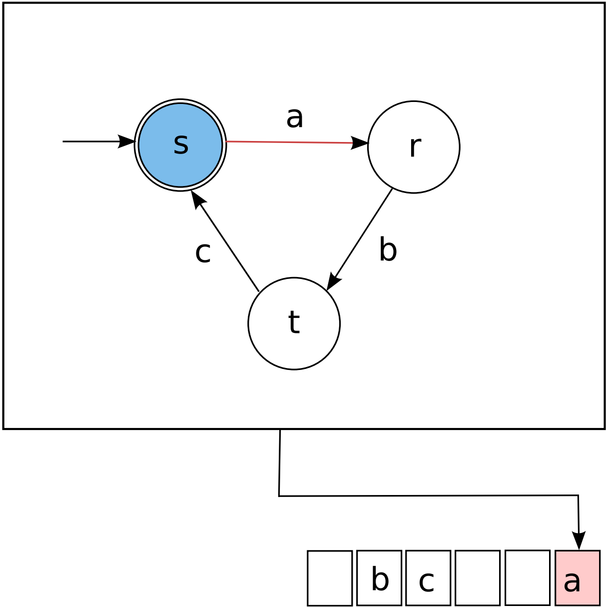

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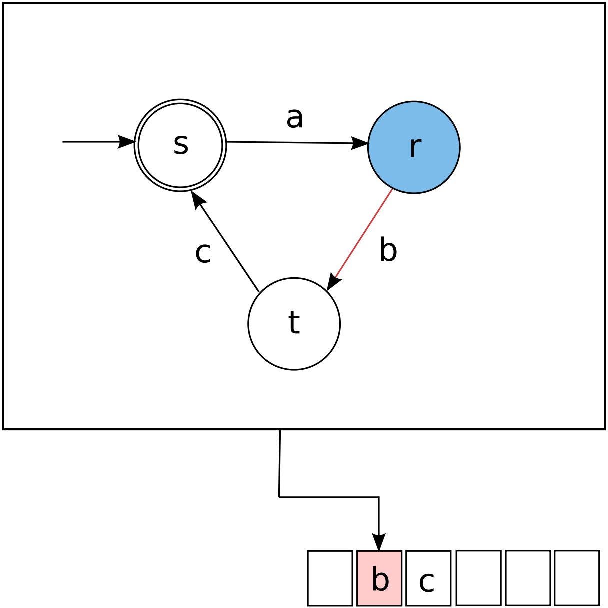

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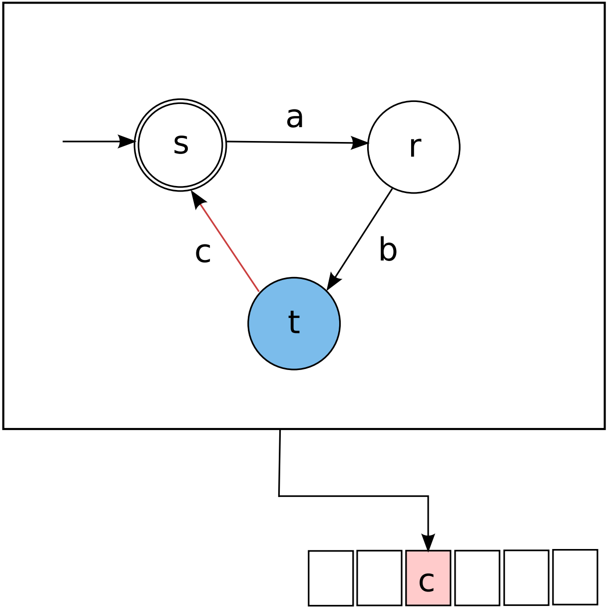

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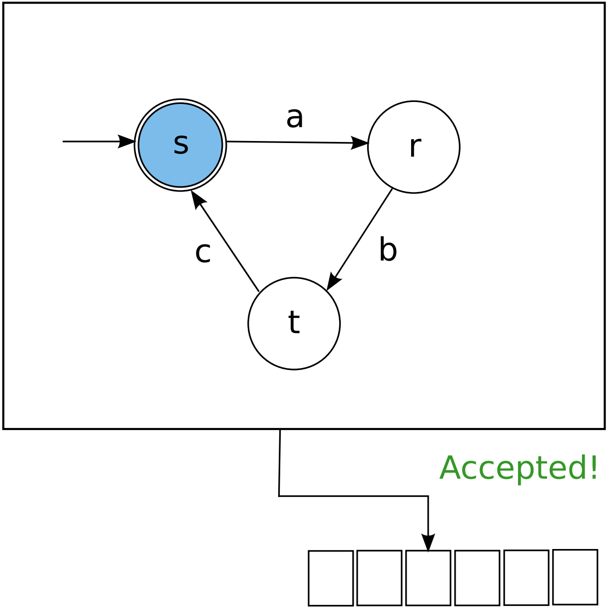

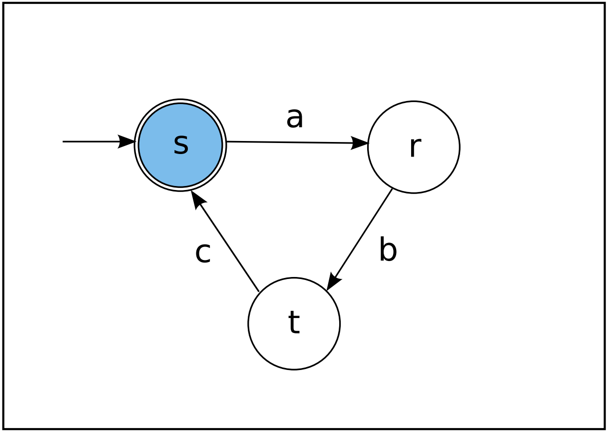

Accepted language: $\{w \in \{a, b, c\}^* : |w|_a = |w|_b = |w|_c\}$

Definition

A *general jumping finite automaton* (*GJFA*) is a quintuple

$$M = (Q, \Sigma, R, s, F)$$

where

- Q is a finite set of *states*;
- Σ is the *input alphabet*;
- R is a finite set of *rules* of the form

$$py \to q \qquad (p, q \in Q, y \in \Sigma^*)$$

- s is the *start state*;
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Definition

If all rules $py \rightarrow q \in R$ satisfy $|y| \leq 1$, then M is a *jumping finite automaton* (*JFA*).

Definition

If $x, z, x', z', y \in \Sigma^*$ such that $xz = x'z'$ and $py \rightarrow q \in R$, then M makes a *jump* from $xpyz$ to $x'qz'$, symbolically written as

$$x\underline{p}yz \curvearrowright x'\underline{q}z'$$

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- $\curvearrowright^n$ intuitively, a sequence of n jumps ($n \geq 0$); mathematically, the nth power of $\curvearrowright$
- $\curvearrowright^*$ intuitively, a sequence of jumps (possibly empty); mathematically, the reflexive-transitive closure of $\curvearrowright$

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Definition

The *language accepted by M*, denoted by $L(M)$, is defined as

$$L(M) = \{uv : u, v \in \Sigma^*, u\underline{s}v \curvearrowright^* \underline{f}, f \in F\}$$

Example

The JFA

$$M = (\{s, r, t\}, \{a, b, c\}, R, s, \{s\})$$

with

$$R = \{sa \rightarrow r, rb \rightarrow t, tc \rightarrow s\}$$

accepts

$$L(M) = \{w \in \{a, b, c\}^* : |w|_a = |w|_b = |w|_c\}$$

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Example

The GJFA

$$H = (\{s, f\}, \{a, b\}, R, s, \{f\}),$$

with

$$R = \{sba \to f, fa \to f, fb \to f\}$$

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$$L(H) = \{a, b\}^* \{ba\} \{a, b\}^*$$

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For instance:

$$bb\underline{s}baa \curvearrowright bb\underline{f}a \quad [sba \to f]$$

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Power of GJFAs and JFAs

Theorem

Let K be an arbitrary language. Then, K is accepted by a JFA only if $K = \mathrm{perm}(K)$.

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When using $pa \rightarrow q$, a can appear on any position. □

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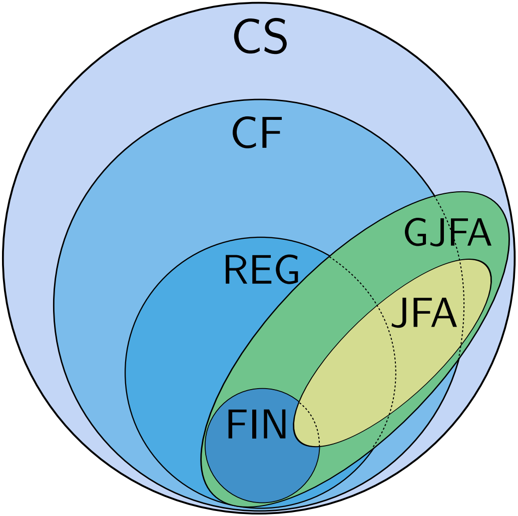
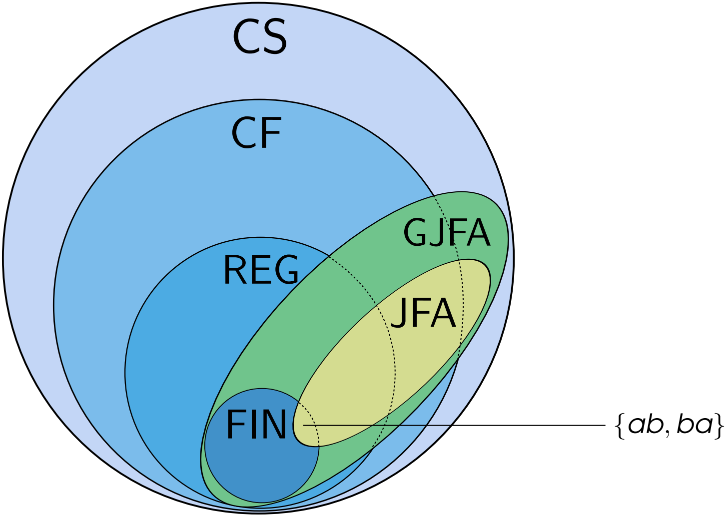
GJFAs are strictly stronger than JFAs.

Proof Idea

The language $\{a, b\}^*\{ba\}\{a, b\}^*$ is accepted by the GJFA from Example #2. □

Relations Between Language Families




CS
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GJFA
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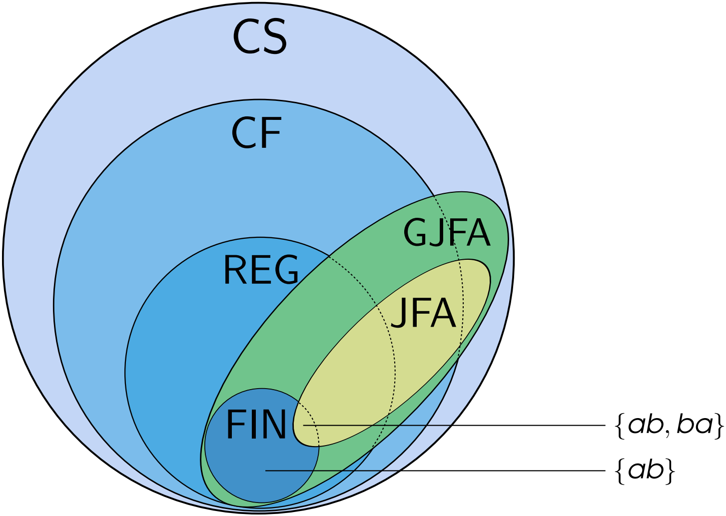
{ab, ba}

Relations Between Language Families



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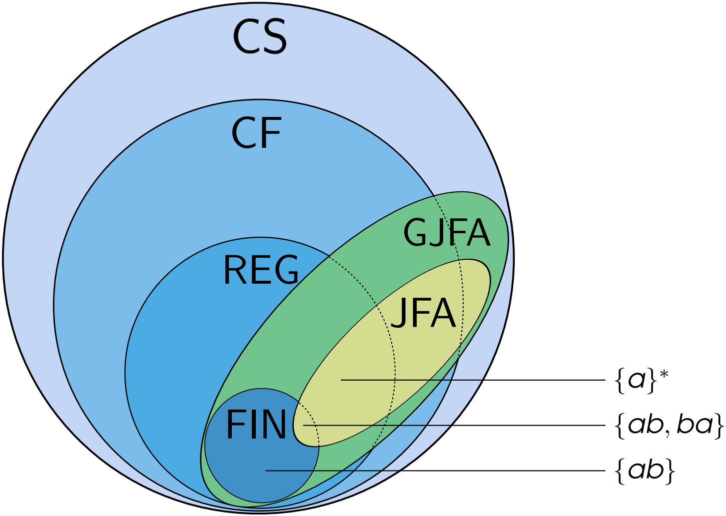

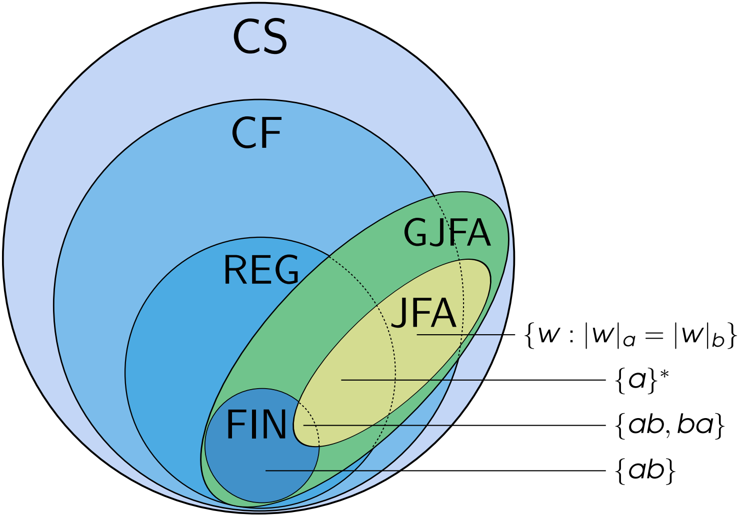
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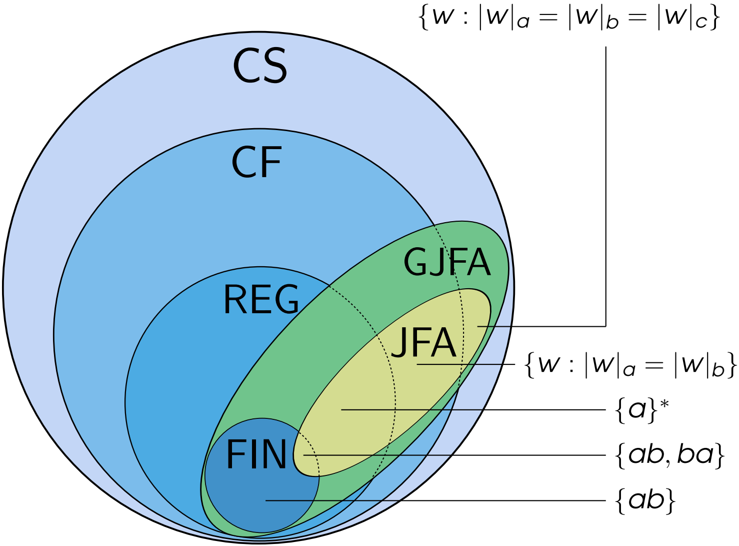

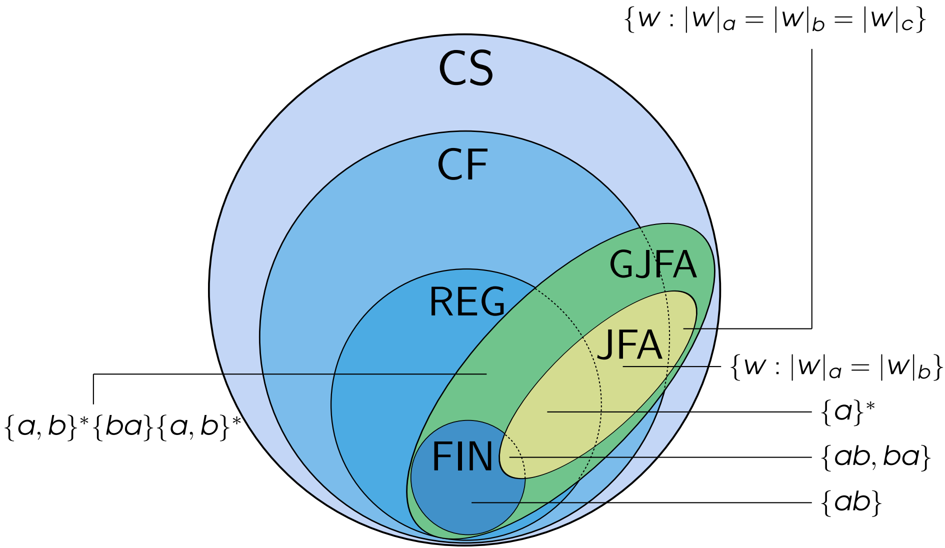
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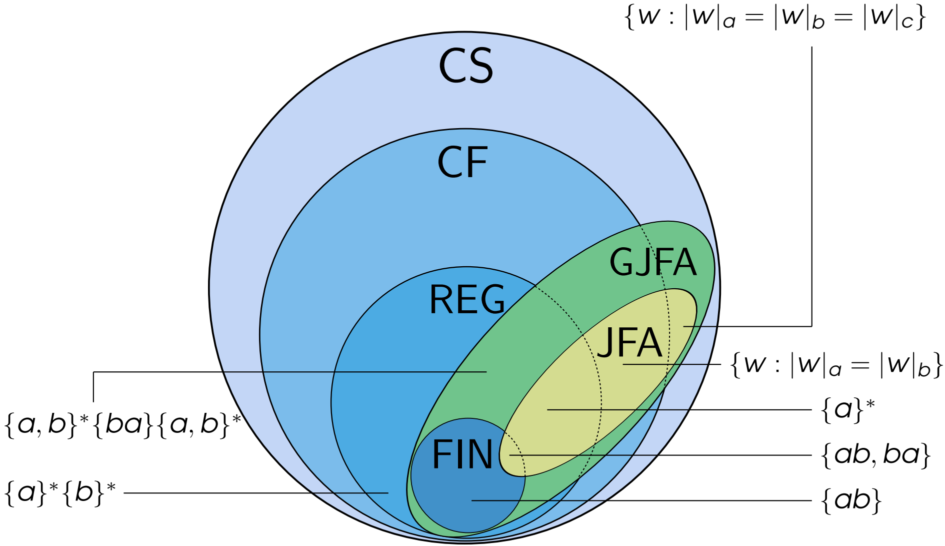

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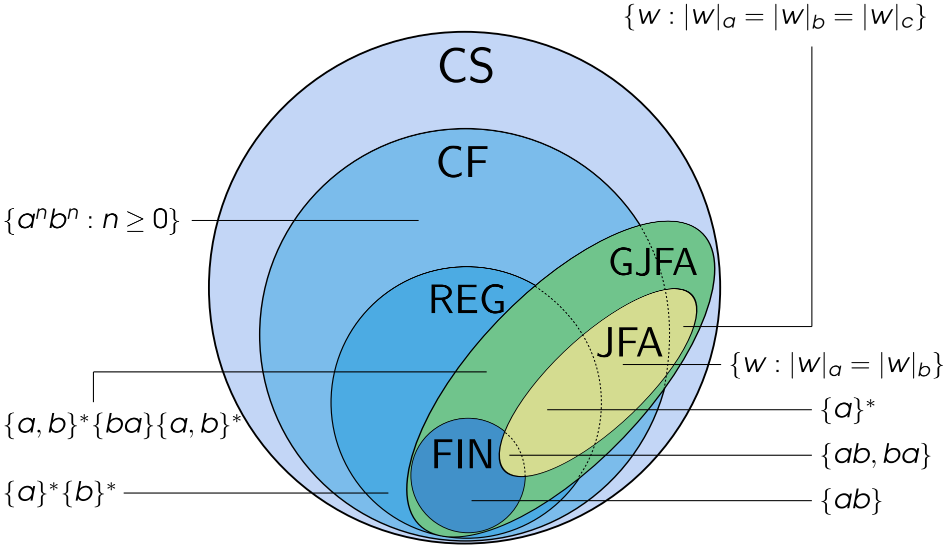

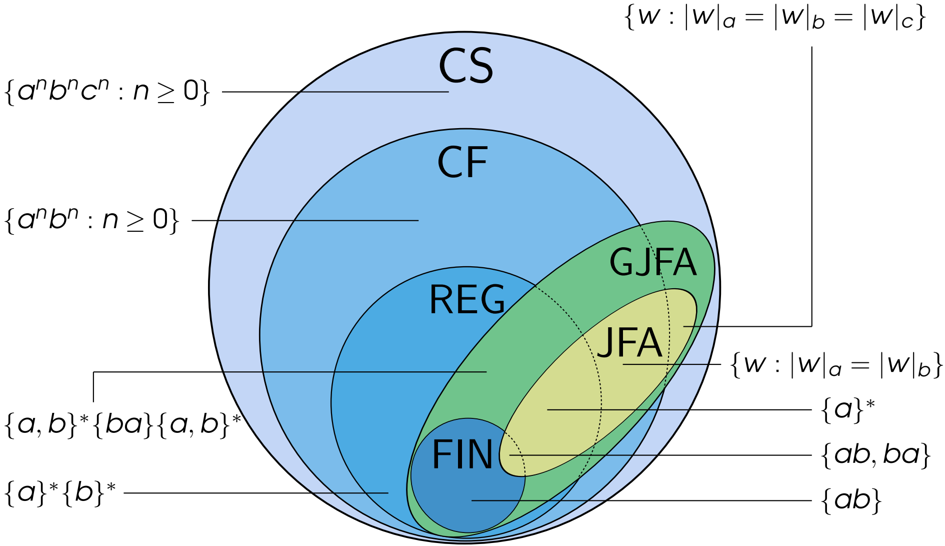
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CS
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By analogy with finite automata:

- removal of ε-*moves* $\quad (p \to q$ and $qa \to r \quad \Rightarrow \quad pa \to r)$
- making JFAs deterministic

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Theorem

Every unary language accepted by a JFA is regular.

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In unary languages, it does not matter where the automaton jumps. □

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Proof Idea

In unary languages, it does not matter where the automaton jumps. □

Corollary

The language of primes

$$\{a^p : p \text{ is a prime number}\}$$

cannot be accepted by any JFA.

Closure Properties

Theorem

JFA *is closed under union.*

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Proof

We have: Two JFAs

- $M_1 = (Q_1, \Sigma_1, R_1, s_1, F_1)$
- $M_2 = (Q_2, \Sigma_2, R_2, s_2, F_2)$ $\quad (Q_1 \cap Q_2 = \emptyset)$

We need: JFA $H = (Q, \Sigma, R, s, F)$ such that $L(H) = L(M_1) \cup L(M_2)$

Construction:

$$\begin{aligned} Q &= Q_1 \cup Q_2 \cup \{s\} && (s \notin Q_1 \cup Q_2) \\ \Sigma &= \Sigma_1 \cup \Sigma_2 \\ R &= R_1 \cup R_2 \cup \{s \to s_1, s \to s_2\} \\ F &= F_1 \cup F_2 \end{aligned}$$

□

Closure Properties – Continued

Theorem

JFA *is not closed under concatenation.*

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Proof

- Consider $K_1 = \{a\}$ and $K_2 = \{b\}$.
- The JFA $M_1 = (\{s, f\}, \{a\}, \{sa \rightarrow f\}, s, \{f\})$ accepts K_1.
- The JFA $M_2 = (\{s, f\}, \{b\}, \{sb \rightarrow f\}, s, \{f\})$ accepts K_2.
- However, there is no JFA that accepts $K_1K_2 = \{ab\}$. $\square$

Closure Properties – Summary

	GJFA	JFA
union	+	+
intersection	−	+
concatenation	−	−
intersection with reg. lang.	−	−
complement	−	+
shuffle	?	+
mirror image	?	+
Kleene star	?	−
Kleene plus	?	−
substitution	−	−
regular substitution	−	−
finite substitution	+	−
homomorphism	+	−
ε-free homomorphism	+	−
inverse homomorphism	+	+

Decidability – Summary

	GJFA	JFA
membership	+	+
emptiness	+	+
finiteness	+	+
infiniteness	+	+

Definition

A GJFA $M = (Q, \Sigma, R, s, F)$ is of *degree* n, where $n \geq 0$, if $py \rightarrow q \in R$ implies that $|y| \leq n$.

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Example

The GJFA $M = (\{s, p, f\}, \{a, b, c\}, R, s, \{f\})$ with

$$R = \{sabc \rightarrow p, pcc \rightarrow f, fa \rightarrow f\}$$

is of degree 3.

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Theorem

$\mathbf{GJFA}_n \subset \mathbf{GJFA}_{n+1}$ *for all* $n \geq 0$

Definition

A GJFA makes a *left jump* from $wxpyz$ to $wqxz$ by $py \rightarrow q$:

$$wx\underline{p}yz \; {}_{l}\!\curvearrowright \; w\underline{q}xz$$

where $w, x, y, z \in \Sigma^*$.

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A GJFA makes a *right jump* from $wpyxz$ to $wxqz$ by $py \rightarrow q$:

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${}_l$**GJFA**	GJFAs using only left jumps
${}_l$**JFA**	JFAs using only left jumps
${}_r$**GJFA**	GJFAs using only right jumps
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Theorem

$_r\mathbf{GJFA} = {}_r\mathbf{JFA} = \mathbf{REG}$

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Proof Idea

- ${}_r\mathbf{JFA} = \mathbf{REG}$ simulating a finite automaton
- ${}_r\mathbf{GJFA} = \mathbf{REG}$ simulating a *general finite automaton* □

Left and Right Jumps – Results

Theorem

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Proof Idea

- $_r\mathbf{JFA} = \mathbf{REG}$ simulating a finite automaton
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Theorem

$_l\mathbf{JFA} - \mathbf{REG} \neq \emptyset$

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- $_r\mathbf{GJFA} = \mathbf{REG}$ simulating a *general finite automaton* □

Theorem

$_l\mathbf{JFA} - \mathbf{REG} \neq \emptyset$

Proof Idea

$$M = (\{s, p, q\}, \{a, b\}, R, s, \{s\})$$

with

$$R = \{sa \to p, pb \to s, sb \to q, qa \to s\}$$

accepts

$$_lL(M) = \{w : |w|_a = |w|_b\}$$ □

Definition

Let $M = (Q, \Sigma, R, s, F)$ be a GJFA. Set

$$
\begin{aligned}
{}^{b}L(M) &= \{w \in \Sigma^* : \underline{s}w \curvearrowright^* \underline{f} \text{ with } f \in F\} && \text{(beginning)} \\
{}^{a}L(M) &= \{uv : u, v \in \Sigma^*, u\underline{s}v \curvearrowright^* \underline{f} \text{ with } f \in F\} && \text{(anywhere)} \\
{}^{e}L(M) &= \{w \in \Sigma^* : w\underline{s} \curvearrowright^* \underline{f} \text{ with } f \in F\} && \text{(end)}
\end{aligned}
$$

Definition

Let $M = (Q, \Sigma, R, s, F)$ be a GJFA. Set

$$
\begin{array}{lcll}
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\end{array}
$$

^{b}**GJFA**	GJFAs starting at the beginning
^{a}**GJFA**	GJFAs starting anywhere
^{e}**GJFA**	GJFAs starting at the end
^{b}**JFA**	JFAs starting at the beginning
^{a}**JFA**	JFAs starting anywhere
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^{b}**GJFA**	GJFAs starting at the beginning
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^{b}**JFA**	JFAs starting at the beginning
^{a}**JFA**	JFAs starting anywhere
^{e}**JFA**	JFAs starting at the end

Observations:

- ${}^{a}L(M) = L(M)$
- ^{a}**GJFA** $=$ **GJFA** and ^{a}**JFA** $=$ **JFA**

A Variety of Start Configurations – Results

Theorem

${}^{a}\mathbf{JFA} \subset {}^{b}\mathbf{JFA}$

Theorem

${}^{a}\mathbf{JFA} \subset {}^{b}\mathbf{JFA}$

Proof Idea

The JFA

$$M = (\{s, f\}, \{a, b\}, \{sa \rightarrow f, fb \rightarrow f\}, s, \{f\})$$

satisfies ${}^{b}L(M) = \{a\}\{b\}^{*}$ $(\{a\}\{b\}^{*} \notin {}^{a}\mathbf{JFA})$. □

Theorem

${}^{a}\mathbf{JFA} \subset {}^{b}\mathbf{JFA}$

Proof Idea

The JFA

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Theorem

${}^{a}\mathbf{GJFA} \subset {}^{b}\mathbf{GJFA}$

Theorem

^{a}**JFA** $\subset$ ^{b}**JFA**

Proof Idea

The JFA

$$M = (\{s, f\}, \{a, b\}, \{sa \rightarrow f, fb \rightarrow f\}, s, \{f\})$$

satisfies $^{b}L(M) = \{a\}\{b\}^{*}$ ($\{a\}\{b\}^{*} \notin$ ^{a}**JFA**). □

Theorem

^{a}**GJFA** $\subset$ ^{b}**GJFA**

Theorem

^{e}**GJFA** $=$ ^{a}**GJFA** *and* ^{e}**JFA** $=$ ^{a}**JFA**

Conclusion and Open Problem Areas

- closure properties of **GJFA** (shuffle, Kleene star, Kleene plus, and mirror image)
- other decision problems of **GJFA** and **JFA**, like equivalence, universality, inclusion, or regularity
- the effect of left jumps to the power of JFAs and GJFAs (we only know that ${}_l\mathbf{JFA} - \mathbf{REG} \neq \emptyset$)
- strict determinism
- applications: verification of a relation concerning the number of symbol occurrences (genetics)

Discussion